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Comment on: Curious Properties of Simple Random Walks

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In [1], Ben-Abraham looks at elementary random walks in one dimension, and at the addition of the drift velocities of two walks relative to each other in particular. He demonstrates some curious properties that are reminiscent of mathematical structures in relativity. In probabilistic terminology, this can be reformulated as follows.

Consider a family $(X_i)_{i\in\mathbb{N}}$ of independent, identically distributed (i.i.d.) d-dimensional random variables with finite mean (i.e., each of the coordinate variables $X_{1,i}$, $1\leqslant i\leqslant d$, has finite mean) and finite variance $\Big($i.e., $\sum_{i=1}^{d}\mathbb{V}(X_{1,i})<\infty\Big)$, and let $S := (S_n)_{n\in\mathbb{N}} = \Big(\sum_{i=1}^{n} X_i\Big)_{n\in\mathbb{N}}$ be the associated stochastic process (which is a random walk, if X_1 takes only a discrete number of values with probability one). In [1], for $d=1$, the length of S is defined as the standard deviation of X_1, thus $\ell(S):=\sqrt{\mathbb{V}(X_1)}$, and the velocity of the stochastic process as the expectation of X_1, to be interpreted as the (mean) distance reachable in one unit of time, so $v(S)=\mathbb{E}(X_1)$. $\mathbb{P}(X_1=1)=p=1-\mathbb{P}(X_1=-1)$ gives the ordinary nearest neighbour random walk, with velocity $2p-1$ and variance $4p(1-p)$.

By elementary stochastics, $\mathbb{V}(X_1)=\mathbb{E}(X_1^2)-(\mathbb{E}(X_1))^2$. This can be rewritten as

$$\ell(S^{(v)}) = \left\{1-\frac{v^2}{c^2}\right\}^{1/2} \ell(S^{(0)}) \tag{1}$$

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where the upper index v refers to an ordinary random walk with velocity $v=2p-1$, and $c=1$ is the maximum velocity, the "speed of light", in this model. Therefore, (1) is reminiscent of a length contraction of $S^{(v)}$, as seen from a second walker at rest ($v=0$ corresponds to $p=1/2$, the symmetric walk). Eq. (1) easily generalizes to random walks with jump length x_c (in both directions). For example, the difference walk of two walkers as above, conditioned on the walkers not moving into the same direction, has $x_c=2$. Then, (1) can be interpreted as the associated "length contraction" of one random walk (with fixed jump length) seen from another random walk in motion, conditioned on that they are not at rest relative to each other. In fact, the conditioning is necessary, since, if we also allow for random walks with $\mathbb{P}(X_1=0)=1-q>0$, then $\mathbb{E}(X_1^2)$ becomes smaller by a factor q, while $v(S^{(v)})^2$ becomes smaller by a factor q^2, and (1) breaks down.

In [1], Ben-Abraham remarks that, in higher dimensions, such a behaviour of (simple) random walks cannot be expected, due to the incompatibility of a lattice structure with isotropy of space, and wonders whether (1) is just a curious coincidence or whether there is something deeper hidden behind. This comment is meant to settle these questions.

First, for $d\geqslant 2$, define the velocity of S as

$$v(S) = \left\{\sum_{i=1}^{d}\big(\mathbb{E}(X_{1,i})\big)^2\right\}^{1/2}$$

(this is slightly different from the definition given above, because we only allow for positive velocity; however, for our purposes, there will be no difference, since we are interested in the square of the velocity only, anyway). Moreover, define the length of S as

$$\ell(S) := \left\{\sum_{i=1}^{d}\sigma_i^2\right\}^{1/2},$$

with $\sigma_i^2=\mathbb{V}(X_{1,i})$. If X_1 only takes values $\pm x_c e_i$, $1\leqslant i\leqslant d$, where the e_i denote the standard unit vectors in $\mathbb{R}^d$ and $x_c>0$ is some positive constant, we obtain a scaled version of a d-dimensional random walk (with probability vector $p=(p_1^+, p_1^-, \ldots, p_d^+, p_d^-)$, where the $p_i^\pm$'s are the probabilities that X_1 takes the value $\pm x_c e_i$). Let c be the velocity defined by travelling x_c in one unit of time. Then

$$\ell^2(S^{(v)}) = \sum_{i=1}^{d}\left(\mathbb{E}\big((X_{1,i}^{(p)})^2\big) - \big(\mathbb{E}(X_{1,i}^{(p)})\big)^2\right)$$

$$= x_c^2 \left(1 - \frac{\sum_{i=1}^{d} \left(\mathbb{E}(X_{1,i}^{(p)})\right)^2}{c^2}\right) = \left(1 - \frac{v^2(S^{(v)})}{c^2}\right) \ell^2\left(S^{(0)}\right)$$

so that (1) holds. Since this is true for arbitrary $x_c > 0$, we can also apply it to the difference walk of two such random walkers, conditioned on that they don't stand still relative to each other, and conclude that, from the (conditioned) point of view of one random walker, the other random walker undergoes a length contraction.

A generalization to random walks in d dimensions that take more than $2d$ values is less evident. E.g., already in one dimension, one may consider the random walk generated by X_1 (assumed to take values ± 1 and ± 2 with equal probability 1/4) and the random walk generated by Y_1 (taking values ± 1 with probability 1/8 and ± 2 with probability 3/8). Both these random walks have zero velocity, but, since $\mathbb{V}(X_1) < \mathbb{V}(Y_1)$, the first random walker undergoes a length contraction with respect to the second walker. So, (1) does not hold even though there is a maximum velocity.

Another indication that one should not overstretch the interpretation as a length contraction comes from the following observation. For $0 \leqslant r \leqslant c$ (c the maximum velocity), let X_1 be $\mathcal{N}\left(r, \{1 - \frac{r^2}{c^2}\}^{1/2}\right)$–distributed. Then, (1) is trivially true even though the individual jump size is unbounded.

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REFERENCE

1. S. I. Ben-Abraham, Curious properties of simple ramdom walks, *J. Stat. Phys.* **73**:441–445 (1993).